MATH 2040 Lecture 11 (Oct 17, 2016)

§ Geometry of vector spaces

- want to talk about length, angle, ...

Vector space : $(V, +, \cdot)$ $\boxed{\mathbb{F} = \mathbb{R} \text{ or } \mathbb{C}}$
(/ $\mathbb{F}$)

Def$^{\underline{n}}$: An inner product on V is a "function"

$$\langle\, ,\rangle : V \times V \longrightarrow \mathbb{F}$$

s.t. (1) (Linear in 1st slot)

$$\langle a\vec{u} + b\vec{v}, \vec{w}\rangle = a\langle \vec{u}, \vec{w}\rangle + b\langle \vec{v}, \vec{w}\rangle$$

$$\forall\, \vec{u}, \vec{v}, \vec{w} \in V, \quad \forall\, a, b \in \mathbb{F}$$

(2) (Conjugate symmetry)

$$\langle \vec{u}, \vec{v}\rangle = \overline{\langle \vec{v}, \vec{u}\rangle}$$ $\leftarrow \mathbb{C}$ conjugate

$$\forall\, \vec{u}, \vec{v} \in V$$

(3) (Positivity)

$$\langle \vec{u}, \vec{u}\rangle \geq 0 \quad \& \quad \text{"="} \iff \vec{u} = \vec{0}.$$

(if $\mathbb{F} = \mathbb{C}$, $\uparrow$ is $\mathbb{R}$ and non-neg.)

Name: $(V, \langle , \rangle)$ is a real/complex inner product space.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\mathbb{R} \qquad \mathbb{C}$

Lemma: (a) (Conjugate linear in 2^{nd} slot)

$$\langle \vec{u}, a\vec{v}+b\vec{w} \rangle = \bar{a}\langle \vec{u},\vec{v}\rangle + \bar{b}\langle \vec{u},\vec{w}\rangle$$

(b) Suppose $\langle \vec{x}, \vec{y} \rangle = 0$ for all $\vec{y} \in V$.

Then, $\vec{x} = \vec{0}$. (non-degeneracy)

(Caution: $\langle \vec{x}, \vec{y} \rangle = 0$ for some $\vec{y} \in V$

$\not\Rightarrow \vec{x} = \vec{0}$.)

Proof of Lemma:

(a)
$$\langle \vec{u}, a\vec{v}+b\vec{w} \rangle \overset{(2)}{=} \overline{\langle a\vec{v}+b\vec{w}, \vec{u} \rangle}$$

$$\overset{(1)}{=} \overline{a\langle \vec{v},\vec{u}\rangle + b\langle \vec{w},\vec{u}\rangle}$$

$$= \bar{a}\,\overline{\langle \vec{v},\vec{u}\rangle} + \bar{b}\,\overline{\langle \vec{w},\vec{u}\rangle}$$

$$\overset{(2)}{=} \bar{a}\langle \vec{u},\vec{v}\rangle + \bar{b}\langle \vec{u},\vec{w}\rangle$$

(b) Take $\vec{y} = \vec{x}$. Then by assumption

$$\langle \vec{x}, \vec{x} \rangle = 0 \overset{(3)}{\Longrightarrow} \vec{x} = \vec{0}.$$

$\square$

Examples (finite dimensional) $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

(1) $(\mathbb{R}^n, \cdot)$

"dot product"

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} := \sum_{i=1}^{n} x_i y_i$$

E.g. $\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = ac + bd$

(2) $(\mathbb{C}^n, \cdot)$

(complex) dot product

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} := \sum_{i=1}^{n} x_i \overline{y_i}$$

E.g. $\begin{pmatrix} 1 \\ i \end{pmatrix} \cdot \begin{pmatrix} i \\ 1 \end{pmatrix} = 1 \cdot (-i) + i \cdot (1) = 0$

(3) $(M_{n \times n}(\mathbb{R}), \langle , \rangle)$

$$\langle A, B \rangle := \operatorname{tr}(B^t A)$$

(Frobenius inner product)

(4) $(M_{n \times n}(\mathbb{C}), \langle , \rangle)$

$$\langle A, B \rangle := \operatorname{tr}(\overline{B}^t A) = \operatorname{tr}(B^* A)$$

where $B^* = \overline{B^t}$ (conjugate transpose / adjoint)

E.g. $\begin{pmatrix} 1 & i \\ 1 & 0 \end{pmatrix}^* = \begin{pmatrix} 1 & 1 \\ -i & 0 \end{pmatrix}$

Examples (infinite dimensional)

(a) $V = C([0,1]) = \{ f : [0,1] \to \mathbb{R} \text{ cts} \}$ $\mathbb{F} = \mathbb{R}$

$$\langle f, g \rangle_{L^2} := \int_0^1 f(x) g(x) \, dx$$

L^2- inner product

(Ex: check it is inner product, esp. (3))

(b) $V = C_{\mathbb{C}}([0,2\pi]) = \{ f : [0,2\pi] \to \mathbb{C} \text{ cts} \}$

$$\langle f, g \rangle_{L^2} := \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{g(x)} \, dx$$

(comes up in Fourier analysis)

E.g. $\langle \sin t, \cos t \rangle_{L^2} = \frac{1}{2\pi} \int_0^{2\pi} \sin t \cos t \, dt$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \sin 2t \, dt$$

$$= \frac{1}{2\pi} \left(-\frac{1}{4} \cos 2t \right) \Big|_0^{2\pi} = 0$$

Def$^{\underline{n}}$: (Length / norm) $(V, \langle , \rangle)$ inner product space

$$\|\vec{x}\| := \sqrt{\underbrace{\langle \vec{x}, \vec{x} \rangle}_{(3) \Rightarrow \langle \vec{x}, \vec{x} \rangle \geq 0}} \geq 0$$

Prop: (a) $\|c\vec{x}\| = |c| \cdot \|\vec{x}\|$

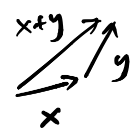
(b) $\|\vec{x}\| = 0 \iff \vec{x} = \vec{0}$.

(c) Cauchy-Schwarz inequality

$$\boxed{|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \, \|\vec{y}\|}$$

(d) Triangle inequality

$$\boxed{\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|}$$



Proof: (a) $\|c\vec{x}\|^2 := \langle c\vec{x}, c\vec{x} \rangle$

$$= c\bar{c} \langle \vec{x}, \vec{x} \rangle$$

$$= |c|^2 \|\vec{x}\|^2$$

Then take $\sqrt{}$ on both sides.

(b) Trivial. (by (3) for inner product)

(c) E.g. $\left|\begin{pmatrix}a\\b\end{pmatrix}\cdot\begin{pmatrix}c\\d\end{pmatrix}\right| \le \left\|\begin{pmatrix}a\\b\end{pmatrix}\right\| \left\|\begin{pmatrix}c\\d\end{pmatrix}\right\|$

$(\mathbb{R}^2, \cdot)$

$$(ac+bd)^2 \le (a^2+b^2)(c^2+d^2).$$

$(C^2([0,1]), \langle , \rangle_{L^2})$: $$\left|\int fg\right| \le \left(\int f^2\right)^{1/2}\left(\int g^2\right)^{1/2}$$

Conside any $c \in \mathbb{F}$,

$$\begin{aligned} 0 &\le \|\vec{x} - c\vec{y}\|^2 \\ &:= \langle \vec{x} - c\vec{y}, \vec{x} - c\vec{y} \rangle \\ &= \langle \vec{x}, \vec{x} \rangle - c\langle \vec{y}, \vec{x} \rangle - \bar{c}\langle \vec{x}, \vec{y} \rangle \\ &\quad + |c|^2 \langle \vec{y}, \vec{y} \rangle \end{aligned}$$

Take $\boxed{c = \dfrac{\langle \vec{x}, \vec{y} \rangle}{\langle \vec{y}, \vec{y} \rangle}}$. (Assume $\vec{y} \ne \vec{0}$)

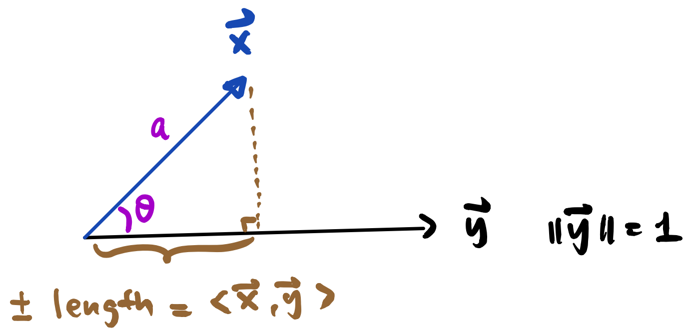
$$\begin{aligned} 0 &\le \|\vec{x}\|^2 - \frac{\langle \vec{x}, \vec{y} \rangle \overline{\langle \vec{x}, \vec{y} \rangle}}{\|\vec{y}\|^2} - \cancel{\frac{\overline{\langle \vec{x}, \vec{y} \rangle} \langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|^2}} \\ &\quad + \cancel{\frac{|\langle \vec{x}, \vec{y} \rangle|^2}{\|\vec{y}\|^4}} \|\vec{y}\|^2 \\ &= \|\vec{x}\|^2 - \frac{|\langle \vec{x}, \vec{y} \rangle|^2}{\|\vec{y}\|^2} \end{aligned}$$

$$\underset{\text{rearrange}}{\Rightarrow} \quad |\langle \vec{x}, \vec{y} \rangle|^2 \le \|\vec{x}\|^2 \|\vec{y}\|^2.$$

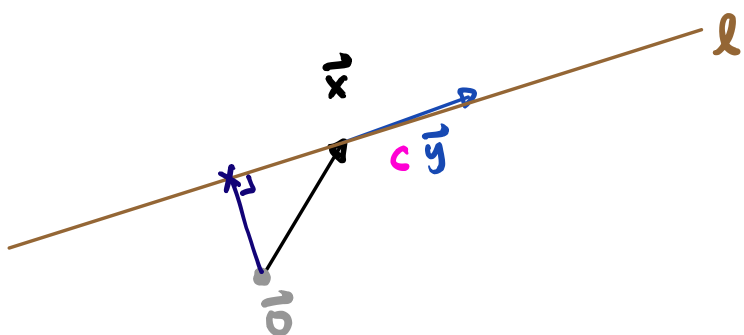
Note: $\boxed{\langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \, \|\vec{y}\| \cos\theta}$ (cosine law)

$(\mathbb{R}^n, \cdot)$

If $\|\vec{y}\| = 1$, then



Back to the proof: (geometric)



Note: $\mathbb{F} = \mathbb{R}$

$$\cos\theta := \frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\| \, \|\vec{y}\|}$$

$\uparrow$ angle between $\vec{x}$ & $\vec{y}$.

$\theta = \frac{\pi}{2}$

$\Downarrow$

$\langle \vec{x}, \vec{y} \rangle = 0$.

Def$^{\underline{n}}$: $\vec{x}$ orthogonal to $\vec{y}$ $\iff$ $\langle \vec{x}, \vec{y} \rangle = 0$.

(ie. $\vec{x} \perp \vec{y}$)

(d) (Triangle ineq): $\|\vec{x}+\vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$.

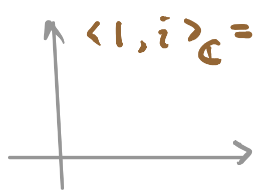
$$
\begin{aligned}
\|\vec{x}+\vec{y}\|^2 &:= \langle \vec{x}+\vec{y}, \vec{x}+\vec{y} \rangle \\
&= \langle \vec{x},\vec{x}\rangle + \langle \vec{y},\vec{x}\rangle + \langle \vec{x},\vec{y}\rangle + \langle \vec{y},\vec{y}\rangle \\
&= \|\vec{x}\|^2 + \overline{\langle \vec{x},\vec{y}\rangle} + \langle \vec{x},\vec{y}\rangle + \|\vec{y}\|^2 \\
&= \|\vec{x}\|^2 + 2\,\mathrm{Re}\langle \vec{x},\vec{y}\rangle + \|\vec{y}\|^2 \\
&\le \|\vec{x}\|^2 + 2\,|\langle \vec{x},\vec{y}\rangle| + \|\vec{y}\|^2 \\
&\le \|\vec{x}\|^2 + 2\,\|\vec{x}\|\,\|\vec{y}\| + \|\vec{y}\|^2 \quad \text{(Cauchy-Schwarz)} \\
&= (\|\vec{x}\| + \|\vec{y}\|)^2 \qquad \square
\end{aligned}
$$

Example:

	$\mathbb{C}$ vs	$\mathbb{R}^2$
	$\mathbb{F} = \mathbb{C}$	$\mathbb{F} = \mathbb{R}$
	$\dim_{\mathbb{C}} = 1$	$\dim_{\mathbb{R}} = 2$
inner product:	$\langle z, w\rangle_{\mathbb{C}} = z\overline{w}$	$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = x_1x_2 + y_1y_2$

Question: What is the concept of "$\perp$"?

$\langle 1, i\rangle_{\mathbb{C}} = 1 \cdot \overline{i} = -i \neq 0$



$1 \perp i$